Rotation Theory for Some Class of Discontinuous Maps of Any Modality

Dissertation submitted to

Visvesvaraya National Institute of Technology, Nagpur

in partial fulfillment of the requirement for the award of the degree

Master of Science

in

Mathematics

by

Ritik Joshi

(MS23MTH022)

under the supervision of

Dr. Sourav Bhattacharya

Department of Mathematics

Visvesvaraya National Institute of Technology

Nagpur - 440010 (India)

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Declaration

I Ritik Joshi, hereby declare that this dissertation titled ”Rotation theory for some class of discontinuous maps of any modality ” is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other institution/University.

Ritik Joshi

Date:

Certificate

This is to certify that the dissertation entitled “Rotation theory for some class of discontinuous maps of any modality” is a bonafide work done by **Ritik Joshi** (Enrollment No. MS23MTH022). The project work has been completed under my guidance and supervision in a satisfactory manner. This work is a part of the requirement for the award of **Master of Science in Mathematics** from Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating the nature and standard of work for the award of Master of Science degree.

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Ritik Joshi

Abstract

In 1971 the celebrated mathematician A.N Sharkovsky came up with a remarkable result to classify periodic orbits of any continuous map from $\mathbb{R} \to \mathbb{R}$ based on their periods. The result was later generalized and sharpened with the introduction of rotation theory. Rotation Theory, originally introduced by Poincare for circle homeomorphisms, was extended to circle maps by Misiurewicz and finally to interval maps by Blokh.

The goal of my thesis is to further extend rotation theory to a class of discontinuous maps that satisfy a specific set of constraints referred to as the *N-type condition*. Given an map f, let M_f denote the point where f reaches its maximum, and m_f denote the point where f reaches its minimum. Also let a_f be the unique fixed point of f. f is then said to satisfy N-type condition if:

1. $M_f \leq a_f \leq m_f$,
2. $f(x) \leq a_f$ for all $x \in [M_f, a_f]$, and $f(x) \geq a_f$ for all $x \in [a_f, m_f]$.

Generalizing rotation theory to such discontinuous maps would help in deriving meaningful insights about their dynamics using minimal information. The theoretical framework developed in this work facilitates the classification of such discontinuous maps into distinct dynamical categories, each with its own characteristic behavior. This study opens a novel line of inquiry, as no existing literature has yet addressed in this direction yet.

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Chapter 1

Introduction

The field of one-dimensional dynamical systems is rich with phenomena arising from the repeated iteration of simple functions. A fundamental goal in this area is to classify and understand the long-term behavior of orbits—whether they stabilize, cycle periodically, or evolve chaotically. To this end, a variety of mathematical tools have been developed, ranging from combinatorial classifications to numerical invariants and analytic conditions.

This thesis explores several interconnected aspects of one-dimensional dynamics, with a particular emphasis on the theory of periodic orbits, rotation numbers, and their generalizations to both continuous and discontinuous maps. The work progresses from classical theorems to modern rotation theory and culminates in new results concerning a class of discontinuous maps that satisfy a structural condition known as the N-type condition.

We begin in **Chapter 3** with *Sharkovsky's Theorem*, a landmark result that establishes an unexpected yet elegant ordering of natural numbers, revealing how the existence of one periodic orbit imposes the existence of many others. In addition to periods, we explore the notion of *patterns*—permutations associated with periodic points—that provide a finer classification scheme and aid in analyzing system complexity, particularly in connection to topological entropy.

Chapter 4 introduces *rotation theory* in the context of circle maps, focusing on the rotation number as a dynamical invariant that captures the average angular displacement of orbits. This invariant refines our understanding of orbit behavior beyond period and lays the foundation for more nuanced analysis of dynamical systems.

Building upon this, **Chapter 5** extends rotation theory to the setting of interval maps through the concept of *over-rotation numbers*. These values generalize the idea of rotation number to maps that are not necessarily defined on the circle, providing a powerful tool to classify periodic

points and infer chaotic behavior. The chapter discusses how over-rotation numbers relate to periodic orbit structure and how they contribute to the estimation of entropy and complexity.

The core contributions of the thesis appear in **Chapter 6**, where we develop a new framework for *rotation theory for a class of discontinuous maps* satisfying an N-type condition. While classical rotation theory has primarily addressed continuous systems, many real-world processes—such as switching systems, digital circuits, and models with abrupt transitions—are inherently discontinuous. To address this, we construct and analyze auxiliary maps that allow us to define an *over-rotation interval* for such discontinuous maps, generalizing the continuous theory.

Applications of this framework are particularly relevant in areas such as:

1. Piecewise-defined economic or biological models, where state-dependent rules introduce discontinuities;

2. Hybrid systems and control theory, where discrete decisions interact with continuous dynamics;

3. Digital signal processing, where discontinuities arise naturally in sampling and switching behavior;

4. Computer graphics and visualization, where discontinuous transformations play a role in image iteration.

Chapter 2

Preliminaries

In this chapter we will revise some of the preliminaries from Real Analysis, Topology, Algebraic Topology and Dynamical Systems.

2.1 Pre-requisites from Real Analysis and General Topology

Definition 2.1.1. [7] (Fixed point) The fixed point for a function f is the point x such that $f(x) = x$.

Definition 2.1.2. [4] (Topological Space) A topological space is defined as a set X equipped with a collection τ of subsets of X that satisfies the following three properties:

- The empty set and the entire set are in τ:
  i.e. $\phi \in \tau$ and $X \in \tau$
- The union of any collection of sets in τ is also in τ.
- The intersection of any finite collection of sets in τ is also in τ.

Definition 2.1.3. [4] (Continuous function) A function $f : X \to Y$ is said to be continuous if every inverse image of an open set is open, where X and Y are topological spaces.

Definition 2.1.4. [4] (Homeomorphism of two topological spaces) A homeomorphism is a special type of function between two topological spaces that preserves the structure of the spaces. Specifically, a function $f : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it satisfies the following conditions:

- Bijective: The function f is one-to-one (injective) and onto (surjective).

- Continuous: The function f is continuous, meaning the preimage of every open set in Y is an open set in X.
- Continuous Inverse: The inverse function $f^{-1} : Y \to X$ is also continuous.

Homeomorphic spaces are topologically equivalent, meaning they have the same topological properties even though they might look different geometrically.

Theorem 2.1.1. [7] (Intermediate Value Theorem) Let f be a continuous function on a closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$.That is, if $f(a) \leq N \leq f(b)$ or $f(b) \leq N \leq f(a)$, then there exists a number c in the interval (a, b) such that:

$$f(c) = N$$

.

Theorem 2.1.2. [7] (Mean Value Theorem) Let f be a function that satisfies the following conditions:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).

Then there exists at least one point c in the interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Definition 2.1.5. [4] (Compact space) A topological space X is sequentially compact if every sequence of points in X has a convergent subsequence converging to a point in X.

Definition 2.1.6. [7] (Modality) The modality of a function f is defined as the total count of its critical points.

Definition 2.1.7. [7] (Critical point) Given a function $f : \mathbb{R} \to \mathbb{R}$, a point $x = c$ is a critical point if one of the following conditions holds:

- $f'(c) = 0$ (the derivative of f at c is zero).
- $f'(c)$ does not exist (the derivative of f at c is undefined).

2.2 Pre-requisite from Dynamical Systems

A dynamical system is a field of mathematics focused on understanding the evolution of systems over time. To formally define a dynamical system, two components are essential:

- A space representing the system itself, typically given as a topological space (X, τ)
- A rule that governs how the system evolves, usually described by a continuous function $f : X \to Y$

Example 2.2.1. *Example of a dynamical system arises in modeling population growth. Consider a simplified scenario where the population of a species evolves in discrete time steps, with growth influenced by both reproduction and limited resources. This situation can be described by the logistic map:*

$$x_{n+1} = rx_n(1 - x_n)$$

Here, x_n represents the population at the n-th time step (scaled between 0 and 1), and r is a parameter that controls the growth rate. The function models the idea that the population grows proportionally to its current size but is constrained by the environment's carrying capacity.

This recursive relation defines a discrete dynamical system. Depending on the value of r, the behavior of the sequence $\{x_n\}$ can vary significantly — from stabilizing at a fixed value, to oscillating between values, or even becoming chaotic. This example demonstrates how simple nonlinear rules can produce rich and complex dynamics.

Another example: A well-known example in evolutionary game theory is the Rock-Paper-Scissors game, where each strategy defeats one and is defeated by another. The standard payoff matrix is:

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Let $\vec{x} = (x_1, x_2, x_3)^T$ represent the population state, where x_i is the proportion of individuals using strategy i and $x_1 + x_2 + x_3 = 1$. The replicator dynamics are given by:

$$\frac{dx_i}{dt} = x_i \left[(A\vec{x})_i - \vec{x}^T A\vec{x} \right], \quad i = 1, 2, 3$$

In this model, the system has a fixed point at the center of the simplex, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, where

all strategies are equally represented. This point is neutrally stable, and the trajectories form closed orbits around it. As a result, the proportions of strategies cycle indefinitely, illustrating the persistent dynamic competition among them.

2.2.1 Trajectory and periodic orbit

In this study to denote the family of all real numbers we use $\mathbb{R}$ and to denote the family of all natural numbers by $\mathbb{N}$.

In the context of dynamical systems, a trajectory (or orbit) refers to the path that a point in the system follows through its state space as the system evolves over time. This concept is crucial in understanding the behavior of dynamical systems, whether they are described by differential equations, discrete maps, or other types of dynamical rules.

Definition 2.2.1. [3] (Trajectoy) The series $\{u_n\}$, established iteratively as $u_{n+1} = h(u_n)$ within the function h, represents the trajectory originating from the initial point u_0 under the influence of the function h.

Definition 2.2.2. [6] (Periodic Orbit) For any x in X, if there exists a positive integer p such that $h^p(x) = h$, then x is considered a periodic point of period p. The set $\{x, h(x), h^2(x), h^3(x), ..., h^{p-1}(x)\}$ is termed a periodic orbit or cycle of period p.

Definition 2.2.3. [6] (Periodic point) A periodic point in a dynamical system is a point that returns to its original position after a fixed number of iterations of a given function. Specifically, for a function f and a point x, x is a periodic point if there exists a positive integer n such that $f^n(x) = x$, where f^n denotes the function f applied n times. The smallest such n is called the period of the point. In other words, a periodic point is a state that the system revisits exactly after n iterations.

Example 2.2.2. *Fuctions like $F(x) = x$ and $F(x) = -x$ exhibit the unique characteristic that every point is periodic under their iterations. However, these functions are uncommon among dynamical systems.*

Functions with hyperbolic periodic points are commonly found in many dynamical systems and offer the most straightforward types of periodic behavior to study.

2.2.2 Hyperbolicity

Definition 2.2.4. [6] (Hyperbolicity) A periodic point q with a prime period m is considered hyperbolic if the condition

$$|(h^m)'(q)| \neq 1$$

is satisfied. The quantity $(h^m)'(q)$ is known as the multiplier of the periodic point.

Example 2.2.3. *Example of hyperbolic point: Consider*

$$h(y) = \frac{1}{2}(y^3 + y)$$

The function has three fixed points at $y = 0, 1$ *and* -1.
Observe that the derivative at these points is $h'(0) = \frac{1}{2}$ *and* $h'(\pm 1) = 2$.
Consequently, all these fixed points are hyperbolic.

Definition 2.2.5. [6] (Attracting Point) A hyperbolic periodic point p of period n is called an attracting periodic point (also known as an attractor or sink) if $|(f^n)'(p)| < 1$.

Proposition 2.2.1. [6] If q is a hyperbolic fixed point with $|g'(q)| < 1$, then there exists an open interval U around q such that for any $x \in U$, the iterates of x under g converge to q. In other words,

$$\lim_{n\to\infty} g^n(x) = q$$

Proof. Let $g \in C'$. Let $\epsilon > 0$ and $x \in [q - \epsilon, q + \epsilon]$
$\implies$ g' is a continuous and bounded function.
$\therefore |g'(x)| < A$, where $A \in \mathbb{R}$
Given that $|g'(q)| < 1$
$\therefore$ By neighbourhood property of continuous function,
$|g'(x)| < 1$
$\therefore |g'(x)| < A < 1$
Then, by mean value thoerem, there exist $h \in [p - \epsilon, p + \epsilon]$ such that
$|g(x) - q| = |g(x) - g(p)| \leq g'(h) \cdot |x - q| < |x - q| \leq \epsilon$
Hence, $g(x) \in [p - \epsilon, p + \epsilon]$
Similary, $|g^2(x) - q| \leq A^2 \cdot |x - q|$
In general, $|g^n(x) - q| \leq A^n \cdot |x - q|$
$\therefore g^n(x) \to q$ as $n \to \infty$ (since $A < 1$) □

Remark 2.2.1. *Attracting periodic points of period n therefore possess neighborhoods that are mapped into themselves by g^n. This type of neighborhood is referred to as the Local Stable Set.*

Definition 2.2.6. [6] (Repelling Point) A fixed point q with $|g'(q) > 1|$ is called a repelling fixed point (a repeller or source).

Proposition 2.2.2. [6] If q is a hyperbolic fixed point with $|g'(q) > 1|$, then there exists an open interval U around q such that, for any $x \in U$ with $x \neq q$, there is some $k > 0$ for which $g^k(x) \notin U$.

Proof. Let $g \in C'$. Let $\epsilon > 0$ and $x \in [q - \epsilon, q + \epsilon]$
Given that $|g'(x)| > 1$
We choose $A \in \mathbb{R}$ such that $|g'(x)| > A > 1$
Bye mean value theorem there exist $h \in [q - \epsilon, q + \epsilon]$ such that,
$|g(x) - q| = |g(x) - g(p)| = g'(h) \cdot |x - q| > A \cdot |x - q|$
$\therefore |g^n(x) - q| > A^n \cdot |x - q|$
As $n \to \infty$, $a^n \to \infty$, since $A > 1$
$\therefore$ There does not exist any k for which $g^k(x) \in [q - \epsilon, q + \epsilon]$ □

2.2.3 Topological conjugacy

Definition 2.2.7. [6] (Topological conjugacy) Two functions f and g are said to be topologically conjugate to each other if there exist a bijective map ψ such that

$$g = \psi^{-1} \circ f \circ \psi$$

or

$$\psi \circ g = f \circ \psi$$

Topological Conjugacy is a concept in dynamical systems theory that describes a strong form of equivalence between two dynamical systems. Two systems are topologically conjugate if their behavior can be transformed into each other via a continuous, invertible change of coordinates.

Topological conjugacy is a powerful tool in dynamical systems because it allows mathematicians to classify systems up to a very strong form of equivalence. If two systems are topologically conjugate, understanding the dynamics of one immediately provides understanding of the other.

This concept is especially useful in studying chaotic systems, where small differences in initial conditions can lead to vastly different behaviors, yet the overall structure of their chaos can be understood through conjugacy.

In summary, topological conjugacy provides a rigorous way to determine when two dynamical systems are essentially the same from the perspective of their topological dynamics.

Example 2.2.4. *Consider two dynamical systems defined on the real line:*

$f(x) = 2x$ *on* $\mathbb{R}$

$g(y) = 3y$ *on* $\mathbb{R}$

The homeomorphism is $h : \mathbb{R} \to \mathbb{R}$ *such that,*

$$h(x) = x^{\frac{\log 3}{\log 2}}$$

To verify that h *is indeed a homeomorphism, note that it is a continuous, monotonic function (since* $\frac{\log 3}{\log 2} > 0$ *and its inverse* $h^{-1}(y) = y^{\frac{\log 2}{\log 3}}$ *is also continuous and monotonic. Therefore, h is a homeomorphism, and we have:*

$$h(f(x)) = h(2x) = (2x)^{\frac{\log 3}{\log 2}} = 2^{\frac{\log 3}{\log 2}} x^{\frac{\log 3}{\log 2}} = 3x^{\frac{\log 3}{\log 2}} = g(h(x))$$

So $h \circ f = g \circ h$*, showing that* f *and* g *are topologically conjugate.*

2.2.4 Structural Stability

Structural stability is a fundamental idea in the study of dynamical systems. It refers to the property that the system's overall qualitative behavior remains unaffected by small changes or disturbances. In other words, a system is considered structurally stable if slight modifications (within a defined class of perturbations) produce a system that behaves in a qualitatively similar way to the original.

To express this more precisely, let us consider a dynamical system defined by a function $f : X \to X$. The system is structurally stable if, for every function $g : X \to X$ that is sufficiently close to f in a suitable topology, there exists a homeomorphism $h : X \to X$ such that g is topologically conjugate to f. This means that h is a continuous, invertible mapping with a continuous inverse, and it satisfies the relation:

$$h \circ f = g \circ h.$$

This condition ensures that the dynamical structures of f and g are essentially the same, even though the systems themselves may differ slightly.

2.2.5 Chaos

Chaos is a concept in dynamical systems describing systems that exhibit sensitive dependence on initial conditions, where small differences in initial conditions lead to vastly different outcomes. Chaotic systems, while deterministic, appear random and unpredictable due to their inherent complexity.

- Sensitivity to Initial Conditions: Also known as the butterfly effect, this property means that small differences in the initial state of the system can lead to vastly different outcomes.
- Topological Mixing: Any given region of the system's phase space will eventually overlap with any other region, indicating that the system evolves in a way that leads to a thorough mixing of phase space.
- Dense Periodic Orbits: Within the chaotic region, there are an infinite number of periodic orbits, which are closely packed.

2.2.6 Topological Entropy

Topological entropy quantifies the exponential growth rate of distinguishable orbits as the system evolves, reflecting the system's unpredictability and chaotic behavior.

Topological entropy is a concept used to measure the complexity of a dynamical system. It provides a numerical value that describes how chaotic the system is. The higher the topological entropy, the more complex and chaotic the system's behavior.

The number and complexity of periodic orbits contribute directly to topological entropy. A greater number of distinct periodic orbits, especially those with longer periods, increases the system's entropy.

2.2.7 The Schwarzian Derivative

Definition and Basic Properties

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^3 function with $f' \neq 0$. The **Schwarzian derivative** of f at $x \in \mathbb{R}$ is defined as:

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Example 2.2.5. *For the sine function:*

$$S(\sin x) = -1 - \frac{3}{2}\tan^2 x,$$

which is negative for all $x \in \mathbb{R}$.

Key Theorems

Theorem 2.2.1. [8] Let $P(x)$ be a polynomial. If all the roots of $P'(x)$ are real and distinct, then $S(P)(x) < 0$ for all $x \in \mathbb{R}$.

Proof. Using the product form of $P'(x)$, one derives expressions for $P''(x)$ and $P'''(x)$, leading to an expression for $S(P)(x)$ that is clearly negative. □

Theorem 2.2.2. [8] If $S_f < 0$ and $S_g < 0$, then $S(f \circ g)(x) < 0$ for all x.

Proof. By the chain rule:

$$S(f \circ g)(x) = S_f(g(x)) \cdot (g'(x))^2 + S_g(x),$$

which is negative under the given conditions. □

Corollary 2.2.1. *If* $S_f < 0$*, then* $S_{f^n} < 0$ *for all* $n > 0$.

Dynamical Implications

Theorem 2.2.3. [8] If $S_f < 0$ and f' has a local extremum at c, then the signs of $f'(c)$ and $f'''(c)$ must be opposite.

Theorem 2.2.4. [8] Let $f \in C^3(\mathbb{R})$ with $S_f < 0$. If there exist points $x_1 < x_2 < x_3$ such that

$$0 < f'(x_2) \leq f'(x_1) = f'(x_3),$$

then f' has a local minimum in (x_1, x_3).

Applications

The Schwarzian derivative plays a critical role in the qualitative analysis of one-dimensional dynamical systems. When the Schwarzian derivative is negative, the system exhibits constrained behavior, such as a limited number of attracting periodic orbits.

Theorem 2.2.5 (Singer's Theorem)**.** *If a C^3 map $f : I \to I$ has negative Schwarzian derivative, then the number of attracting periodic orbits of f is at most $n + 2$, where n is the number of critical points of f.*

This result places a strict upper bound on the complexity of attractors and has deep implications in ensuring stability and predictability of system dynamics.

A prominent application is found in **game theory**, specifically in **two-agent, two-strategy congestion games**. In these models, players repeatedly adjust strategies in a way that can be modeled by interval maps. If these maps have negative Schwarzian derivatives, one can derive structural stability and convergence properties of the game dynamics.

In **physics**, such properties are relevant in the analysis of systems with feedback and control constraints, where bounded behavior and stability are desired.

2.3 Applications of Dynamical System to various fields

Dynamical systems, which explore how processes change over time under defined rules, are fundamental across numerous scientific and applied disciplines. Their influence extends into several key areas:

In **physics**, dynamical systems underpin a wide array of phenomena. Classical mechanics, grounded in Newton's laws, uses differential equations to model motion — from the path of planets to the swing of a pendulum or the behavior of oscillating systems. In **quantum mechanics**, the time evolution of quantum states follows the Schrödinger equation, providing probabilistic insights into experimental outcomes. In fields like **thermodynamics** and **statistical mechanics**, dynamical systems help explain how particles behave on a microscopic scale, including how systems reach equilibrium and undergo phase changes. Across these areas, they offer a unifying framework for understanding and predicting physical behavior.

In **economics**, particularly microeconomics, dynamical systems are employed to model complex behaviors such as economic growth, market fluctuations, and business cycles. Tools like stochastic differential equations help economists analyze stock markets, forecast trends, and

craft investment strategies by capturing the inherent unpredictability and volatility of financial systems.

In **astronomy** and **astrophysics**, the motion of celestial objects — including planets, stars, and galaxies — is governed by dynamical laws, often framed as differential equations. Classical mechanics enables predictions of planetary orbits and interactions within solar and stellar systems. In **cosmology**, these principles are extended through Einstein's general relativity to study the dynamics of the entire universe, from cosmic expansion to the role of dark matter and dark energy. Dynamical systems models allow researchers to simulate the evolution of the universe, shedding light on galaxy formation, background radiation, and its ultimate fate.

Climate science also heavily depends on dynamical systems. Climate models integrate variables like temperature, wind, and ocean currents to simulate and predict weather and long-term climate behavior. In **ecology**, similar models help in understanding how energy, nutrients, and species interact within ecosystems. These insights are essential for assessing environmental impacts, managing natural resources, and guiding conservation efforts.

In **neuroscience**, dynamical systems are used to model how neurons interact and how brain activity unfolds over time. Such models help explain complex processes like memory, perception, and motor control, and they contribute to understanding neurological disorders by revealing how disruptions in neural dynamics can lead to disease.

Finally, in **epidemiology**, dynamical systems are central to modeling how infectious diseases spread. By using systems of equations, scientists can predict outbreak patterns and evaluate public health responses such as vaccination strategies or quarantine protocols. These models support efforts to manage disease and improve health outcomes at the population level.

In summary, dynamical systems offer a powerful mathematical lens through which diverse, evolving processes can be understood, predicted, and managed across science and society.

Chapter 3

Sharkovsky's Theorem and Patterns

Introduction

The study of periodic orbits lies at the heart of dynamical systems, particularly in understanding how the presence of one orbit may imply the existence of others. As the number of periodic orbits increases, so does the complexity of the system.

In 1964, Russian mathematician A. N. Sharkovsky introduced a unique ordering of the natural numbers, now known as *Sharkovsky's ordering*. He showed that the existence of a periodic orbit of a certain period forces the existence of orbits with other periods—a result formalized in his famous theorem.

Complementing this idea, the concept of *patterns* provides a structural lens to classify periodic orbits by their underlying permutations. This helps reveal deeper combinatorial relationships between orbits beyond just their periods.

Patterns also play a role in understanding topological entropy—a measure of dynamical complexity. For interval maps, the over-rotation interval $I_f = [r_f, \frac{1}{2}]$ encodes numerical data related to this complexity. Certain patterns, such as over-twist patterns, are instrumental in estimating entropy and identifying chaotic behavior.

This chapter explores Sharkovsky's ordering alongside the theory of patterns, highlighting howe concepts contribute to the broader analysis of dynamical systems.

3.1 Sharkovsky's Ordering of Natural Numbers

According to Sharkovsky's ordering the Natural numbers are arranged in following manner-

$3 \succ_s 5 \succ_s 7 \succ_s 9 \succ_s ...$

$2 \cdot 3 \succ_s 2 \cdot 5 \succ_s 2 \cdot 7 \succ_s 2 \cdot 9...$

$2^2 \cdot 3 \succ_s 2^2 \cdot 5 \succ_s 2^2 \cdot 7 \succ_s 2^2 \cdot 9...$

$\vdots$

$2^n \succ_s 2^{n-1} \succ_s \cdots \succ_s 2 \succ_s 1$

Note: The symbol '$\succ_s$' is called 'is sharper than'. If $a \succ_s b$ then we say that a appears in Sharkovsky's ordering before b.

Let $Sh(k)$ represents set of all positive integers m such that $k \succ_s m$, together with k

Let $P(f)$ be the set of all periods of periodic orbits of map f.

3.2 Sharkovsky's Theorem

Theorem 3.2.1. [2] Suppose $f : [0, 1] \to [0, 1]$ is a continuous function, $m \succ_s n$ and m belongs to $P(f)$, then n belongs to $P(f)$. That is, if f has a periodic orbit of period m and $m \succ_s n$ then f also has periodic orbit of period n.

From the Sharkovsky's Theorem we get a clear idea that if map has a periodic orbit with some period then how many other orbits with another periods this map will have. This information in fact is very crucial for us as it gives us the information about the complexity of the dynamical system.

3.3 Applications of Sharkovsky's Theorem

Sharkovsky's Theorem, a cornerstone of discrete dynamical systems, primarily deals with the existence and ordering of periodic points in one-dimensional maps. It has intriguing applications across various fields due to its implications for understanding periodic behaviors and complex dynamics.

In the realm of population dynamics, where mathematical models are utilized to elucidate the intricacies of population growth and regulation, Sharkovsky's Theorem plays a pivotal role in discerning the potential periodic behaviors that can manifest. Take, for instance, the logistic map, a classic model employed to study population dynamics. By applying Sharkovsky's Theorem to

such models, researchers can ascertain the existence and ordering of periodic solutions, shedding light on the cyclic patterns that populations may exhibit over time. This insight is invaluable for comprehending the oscillatory dynamics inherent in populations, including periods of growth, stability, and collapse. Whether exploring the fluctuations of predator-prey interactions or the dynamics of competing species, understanding the periodic behaviors elucidated by Sharkovsky's Theorem provides crucial insights into the resilience and vulnerability of ecological systems, aiding in informed decision-making for conservation efforts and ecosystem management strategies.

Chaos theory finds myriad applications across various domains: in biology, it aids in understanding population dynamics, genetics, and epidemiology, while in economics, it informs commodity forecasts, business cycle modeling, and economic time-series analysis. Moreover, chaos theory equations are instrumental in simulating communication systems and analyzing the spread of rumors and misinformation on social media platforms, among other contexts.

Sharkovsky's Theorem has emerged as a cornerstone in the realm of dynamical systems theory. This theorem has permeated numerous scientific disciplines, including physics, chemistry, biology, medicine, and economics. Its significance lies in its standardization as a mathematical principle, and Sharkovsky's discovery of a profound numerical ordering has propelled his name to prominence in the scientific community.

3.4 Conclusions from Sharkovsky's Theorem

- If a map f has a periodic orbit prime period then it will definitely have infinitely many periodic orbits with different periods.
- 3 appears in the starting of Sharkovsky's ordering that is 3. If we have dynamical system characterized by a map f and we suppose that the map f has periodic orbit of period 3, then according to Sharkovsky's Theorem the map f will have all other periodic orbits of all natural numbers as their periods and this leads to $CHAOS$ which makes the dynamics of the system most complicated one to study. is the sharpest natural number.
- The converse of this theorem is also true.

Example 3.4.1. *If a map f has a periodic orbit with period 18 then according to Sharkovsky's Theorem we can get a periodic orbit with period 22 for the map f as $18 = 2.9$ which appears before $22 = 2.11$ in Sharkovsky's ordering.*

3.5 Patterns

Let us consider an equivalence relation $'\sim'$ on the collection of all cycles of a function f such that two cycles A and B are said to be equivalent if they have the same underlying permutation.

Definition 3.5.1. [3] (Patterns) The equivalnce classes of this equivalnce relation are called Patterns. Patterns are denoted by capital letters or by π.

Definition 3.5.2. [3] If a periodic orbit A belongs to a pariticular equivalnce class or a pattern then it is said that the cycle A exhibits a pattern π.

Definition 3.5.3. [3] A pattern π forces another pattern ξ, if every continuous map f having a cycle A which exhibits π also has a cycle B which exhibits ξ.

3.6 How to get the Forced patterns

We construct a technique to find all patterns forced by a given pattern π. Consider a periodic orbit P of a map g which exhibits a pattern π. For this we construct a map f. Let the left-end point of P be $x_1 = a$ and right-end point of P be $x_n = b$ and $P = a = x_1, x_2, ..., x_n = b$ where $x_1 < x_2 < ... < x_n$.

Definition 3.6.1. [3] (P-basic Interval) The closure of each component of $[a, b] - P$ is called a P-basic interval.

$\therefore$ Each of the sub-intervals $[a, x_2]$, $[x_2, x_3]$, ..., $[x_{n-1}, b]$ is a P-basic interval.

3.7 P-linear map

We need to get the patterns which are forced by a given pattern. In order to get the forced patterns we define a map called P-linear map.

Definition 3.7.1. [3] (P-linear map) Define a map f such that f coincides with g at all the points of P and which is linear on each P-basic interval. This is called P-linear map.

The patterns of the cycles of the P-linear map f are exactly the patterns forced by a given pattern.

Chapter 4

Rotation Theory for Circle Maps

4.1 Introduction and Motivation

In the study of one-dimensional dynamical systems, Sharkovsky's Theorem offers a compelling framework for classifying periodic orbits by their periods. However, period alone is often a coarse descriptor, insufficient for capturing the finer details of a system's dynamics. A more detailed, albeit complex, alternative involves symbolic permutations of cycles. Rotation theory provides a balanced intermediate tool — a numerical invariant known as the *rotation number* — to measure and classify the long-term average behavior of orbits, especially for systems on the unit circle S^1.

4.2 Rotation Numbers in General Topological Settings

Let X be a compact topological space and $f : X \to X$ a continuous map. Suppose $\phi : X \to \mathbb{R}$ is a continuous observable. Then, for any $x \in X$, the set of all subsequential limits of the sequence

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x))$$

is called the ϕ**-rotation set** of x, denoted $\rho_\phi(x)$. If this set reduces to a single value ρ, we call ρ the **rotation number** of x. The union of all such sets over X defines the ϕ**-rotation set of the map** f, denoted $\rho(f)$.

4.3 Circle Maps and Lifting

Let $f : S^1 \to S^1$ be a continuous **circle map**. Consider the standard covering projection $\pi : \mathbb{R} \to S^1$ given by

$$\pi(t) = e^{2\pi it}.$$

Then there exists a **lifting** $F : \mathbb{R} \to \mathbb{R}$ of f such that

$$f \circ \pi = \pi \circ F.$$

For liftings of degree d, we have $F(x+d) = F(x)+d$ for all $x \in \mathbb{R}$. We are particularly interested in the case $d = 1$, the **degree-one circle maps**, for which liftings preserve the fundamental properties of the original map modulo integers.

4.4 Rotation Number for Circle Maps

Given $F \in \mathcal{L}_1$, the set of degree-one liftings, the **upper and lower rotation numbers** at a point $x \in \mathbb{R}$ are defined respectively as:

$$\bar{\rho}_F(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}, \quad \underline{\rho}_F(x) = \liminf_{n\to\infty} \frac{F^n(x) - x}{n}.$$

If these limits agree, their common value $\rho_F(x)$ is called the **rotation number of** F **at** x.

4.5 Fundamental Theorems

Theorem 4.5.1. [1] If $f : S^1 \to S^1$ has a lifting F, and $f^n(\pi(x)) = \pi(x)$, then there exists an integer k such that:

$$F^n(x) = x + k \quad \text{and} \quad \rho_F(x) = \frac{k}{n}.$$

Theorem 4.5.2. [1] Let $\mathcal{L}_1$ be the collection of degree one lifting map. Where $\mathcal{L}_1' \subset \mathcal{L}_1$ be the set of non-decreasing liftings. If $F \in \mathcal{L}_1'$, then $\rho_F(x)$ exists, is independent of x, and is rational if and only if f has a periodic point.

4.6 Rotation Interval

For general (non-monotonic) maps, the rotation number may vary with x, and its values fill an interval:

Definition 4.6.1. [1] Let $F \in \mathcal{L}_1$ be a lifting. The **rotation interval** of F is defined as:

$$\mathrm{Rot}(F) = \overline{\{\rho_F(x) \mid x \in \mathbb{R}\}}.$$

To compute this interval, we introduce two extremal functions:

$$F_\ell(x) = \inf\{F(y) \mid y \geq x\}, \quad F_u(x) = \sup\{F(y) \mid y \leq x\}.$$

Then the rotation interval satisfies:

$$\mathrm{Rot}(F) = [\rho(F_\ell), \rho(F_u)]\,.$$

4.7 Illustrative Graph of F_ℓ and F_u

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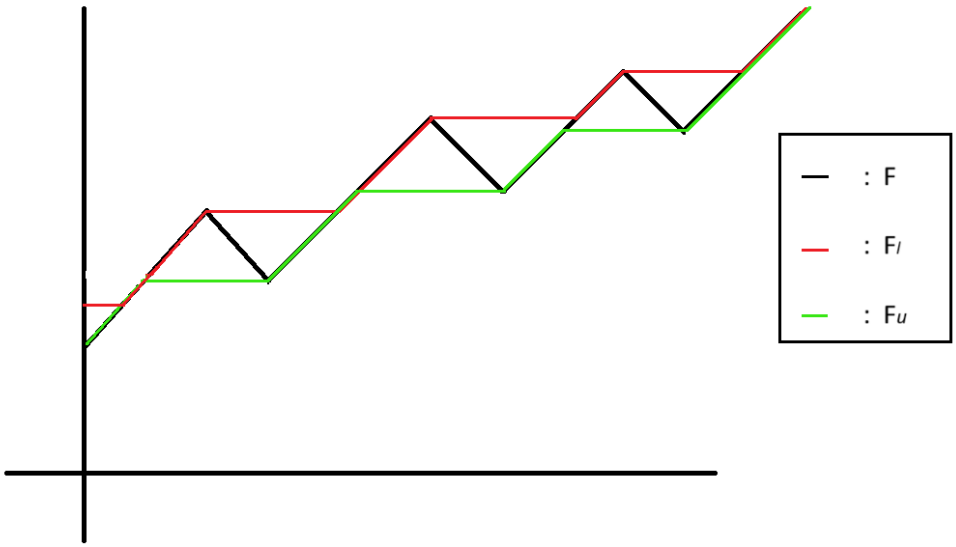

Figure 4.1: F_l and F_u

4.8 Applications

Rotation numbers are central to:

- Classifying orbits and distinguishing between periodic and quasiperiodic behavior
- Predicting synchronization in forced oscillators
- Describing irrational rotations and mode-locking in nonlinear dynamics
- Understanding invariant sets and bifurcations in families of maps

Chapter 5

Rotation Theory For Interval Maps

5.1 Introduction

Rotation theory for interval maps serves as a powerful tool to analyze the long-term behavior of points under iteration. While classical rotation numbers are defined for orientation-preserving circle maps, the theory has been successfully adapted to interval maps through the concept of **over-rotation numbers**. This chapter introduces the definition of over-rotation numbers, discusses their connection to periodic orbits, and explores their role in capturing the dynamical complexity of continuous interval maps.

5.2 Over-Rotation Numbers

Let $f : [0, 1] \to [0, 1]$ be a continuous map. Denote by $\mathrm{Per}(f)$ the set of periodic points of f, and by $\mathrm{Fix}(f) \subseteq \mathrm{Per}(f)$ the set of fixed points. We assume that $\mathrm{Per}(f) \neq \mathrm{Fix}(f)$.

Define a function $\chi : [0, 1] \to \mathbb{R}$ by

$$\chi(x) = \begin{cases} \frac{1}{2} & \text{if } (f(x) - x)(f^2(x) - f(x)) \leq 0, \\ 0 & \text{if } (f(x) - x)(f^2(x) - f(x)) > 0. \end{cases}$$

For any $x \in [0,1]$, consider the sequence

$$\left\{ \frac{1}{n} \sum_{i=0}^{n-1} \chi\big(f^i(x)\big) \right\}_{n=1}^{\infty}.$$

The set of all limit points of this sequence is called the χ-*rotation set* of the point x, denoted by $I_{f,\chi}(x)$.

If $I_{f,\chi}(x) = \{\rho_x\}$ is a singleton, then ρ_x is called the *over-rotation number* of the point x.

The union of over-rotation numbers for all points $x \in [0,1]$ is called the *over-rotation set* of the map f and is denoted by $I_f(\chi)$.

5.3 Over-Rotation Numbers for Periodic Orbits

Let $P \subseteq [0,1]$ be a periodic orbit of the map f with minimal period n. That is, $f^n(x) = x$ for all $x \in P$, and n is the smallest such positive integer.

We define the over-rotation number for this periodic orbit as follows: let m be the number of points $x \in P$ for which

$$(f(x) - x)(f^2(x) - f(x)) \leq 0.$$

Then, the *over-rotation number* of the periodic orbit P is given by

$$\rho(P) = \frac{m}{2n}.$$

Furthermore, the pair $\left(\frac{m}{2}, n\right)$ is called the *over-rotation pair* of the orbit P.

This value provides a rational invariant associated with the orbit, capturing how often the graph of f bends across the identity line $y = x$ within one full period of iteration.

5.4 Over-Rotation Interval

Theorem 5.4.1. [2] (Realization of Over-Rotation Numbers): Let $f : [0,1] \to [0,1]$ be a continuous map. Suppose that f has a periodic orbit with over-rotation number $\frac{p}{q}$, and let $\frac{r}{s} < \frac{p}{q}$. Then f also has a periodic orbit with over-rotation number $\frac{r}{s}$.

This result shows that over-rotation numbers satisfy a density property: the existence of a periodic orbit with a given over-rotation number implies the existence of orbits with all smaller

rational over-rotation numbers.

Definition 5.4.1. [2] (Over-Rotation Interval): The *over-rotation interval* of a continuous interval map f, denoted I_f, is defined as the closure of the set of over-rotation numbers of all its periodic points. That is,

$$I_f = \overline{\{\rho(P) \mid P \text{ is a periodic orbit of } f\}}.$$

It is known that for any continuous map $f : [0,1] \to [0,1]$, the over-rotation interval is of the form

$$I_f = [r_f, \tfrac{1}{2}],$$

where $r_f \geq 0$ is the left endpoint of the interval.

The value $\frac{1}{2}$ is the **maximum possible over-rotation number**. This is because if f has a periodic orbit of any order, then by Sharkovsky's Theorem, it must also possess a periodic orbit of period 2. For such a point with period 2, the over-rotation number is

$$\rho = \frac{1}{2}.$$

Moreover, from the definition of the function χ, we see that this is the greatest value it can attain. Hence, $\frac{1}{2}$ serves as a universal upper bound for over-rotation numbers across all periodic orbits.

The left endpoint r_f serves as a measure of the dynamical richness of the map f. The closer r_f is to $\frac{1}{2}$, the more complex the periodic orbit structure of f is likely to be. In this way, the over-rotation interval encapsulates global information about the map's dynamics through a simple numerical invariant.

Hence, determining r_f becomes a central goal in understanding the rotation-theoretic behavior of continuous interval maps. It provides a compact summary of which over-rotation numbers (and hence which types of periodic dynamics) are permitted by the system.

5.5 Algorithm for Computing the Left Endpoint r_f

We now describe a method to compute the left endpoint of the over-rotation interval $I_f = [r_f, 1/2]$ for a general class of continuous interval maps, regardless of their modality, provided they satisfy specific structural conditions.

Let $f : [0,1] \to [0,1]$ be a continuous map with the following properties:

- $a_f \in [0,1]$ is the unique fixed point of f, i.e., $f(a_f) = a_f$,
- $M_f \in [0,1]$ is a point where f attains a (possibly local) maximum, and $m_f \in [0,1]$ is a point where it attains a (possibly local) minimum,
- These satisfy the ordering:

$$M_f \leq a_f \leq m_f,$$

- The map satisfies the following monotonicity conditions:

$$f(x) \geq a_f \quad \text{for all } x \in [M_f, a_f],$$

$$f(x) \leq a_f \quad \text{for all } x \in [a_f, m_f].$$

These assumptions ensure a structured behavior of the map around its fixed point and are crucial for the algorithmic reduction to classical rotation theory.

Theorem 5.5.1. [2] Let $f : [0,1] \to [0,1]$ be a continuous map satisfying the above conditions. Then there exists a continuous, non-decreasing function $G_f : \mathbb{R} \to \mathbb{R}$, constructed from a suitable conjugate of f, such that the classical rotation number of G_f equals the left endpoint r_f of the over-rotation interval of f.

In particular:

- The function G_f satisfies the lift condition:

$$G_f(x+1) = G_f(x) + 1 \quad \text{for all } x \in \mathbb{R},$$

- The rotation number of G_f is defined as:

$$\rho(G_f) = \lim_{n\to\infty} \frac{G_f^n(x) - x}{n},$$

and is independent of the choice of x,

- The left endpoint of the over-rotation interval of f satisfies:

$$r_f = \rho(G_f).$$

This result enables an algorithmic approach to computing r_f, circumventing the need to exhaustively analyze the full set of periodic orbits of f. The function G_f acts as a bridge between the over-rotation theory of interval maps and the classical rotation number theory developed for monotone degree-one circle maps.

5.6 Graph of the Bimodal Map f

We begin with the continuous interval map $f : [0, 1] \to [0, 1]$ satisfying the following conditions:

$$m_f \leq a_f \leq M_f, \quad f(x) \leq a_f \text{ for } x \in [m_f, a_f], \quad f(x) \geq a_f \text{ for } x \in [a_f, M_f].$$

These conditions define a structured class of maps with a central fixed point a_f, and monotonic behavior around it.

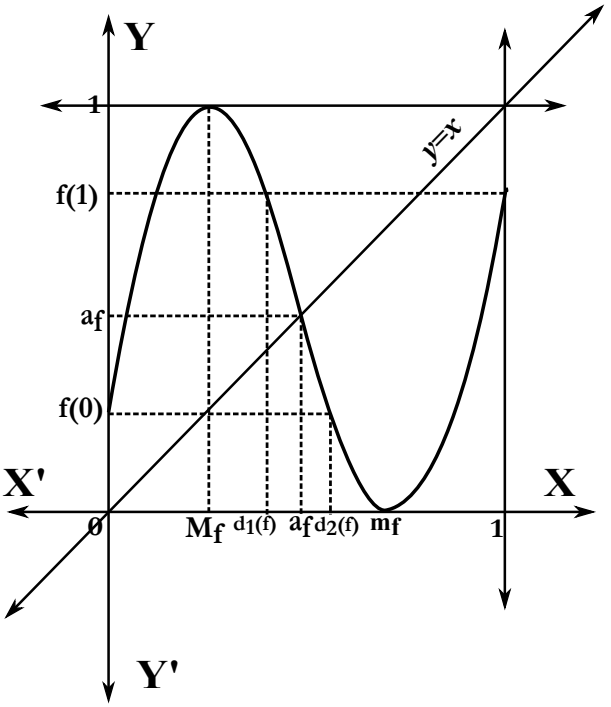

Figure 5.1: An example of a continuous interval map f satisfying the structural conditions.

5.7 Construction and Graph of the Conjugate Map g_f

To symmetrize the behavior of f around its fixed point a_f, we define a conjugate map:

$$g_f = \sigma_f \circ f \circ \sigma_f^{-1}, \quad \text{where } \sigma_f(x) = \begin{cases} x & \text{if } 0 \leq x \leq a_f, \\ a_f + 1 - x & \text{if } a_f < x \leq 1. \end{cases}$$

This transformation reflects the right half of the domain to the left, creating a more symmetric behavior that allows us to define a lift.

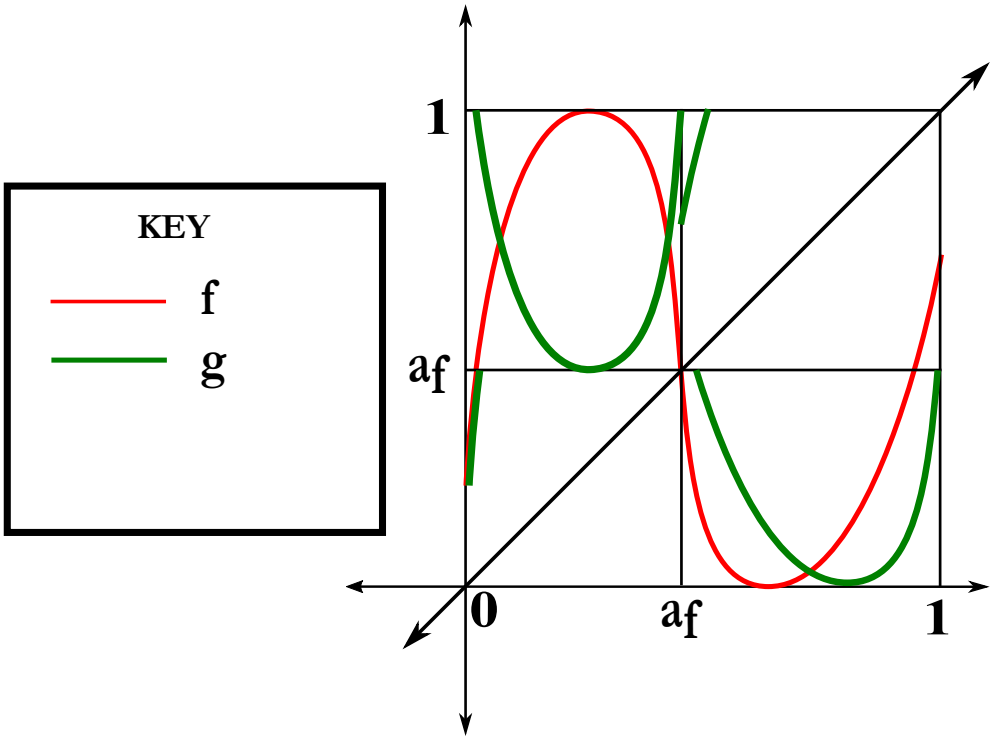

Figure 5.2: The conjugated map g_f obtained via reflection about the fixed point a_f.

5.8 Definition and Graph of the Lifted Map F_f

Using the map g_f, we define a piecewise map $F_f : [0, 1] \to [0, 2]$ as follows:

$$F_f(x) = \begin{cases} g_f(x) & \text{if } x \in [0, a_f], \\ g_f(x) + 1 & \text{if } x \in [a_f, 1] \text{ and } g_f(x) \in [0, a_f], \\ g_f(x) & \text{if } x \in [a_f, 1] \text{ and } g_f(x) \in [a_f, 1]. \end{cases}$$

This definition ensures that the lift increases continuously and forms a candidate for constructing a monotone degree-one map.

5.9 Extension of F_f to the Real Line

We extend the map F_f to the real line to create a degree-one lift:

$$F_f(x+1) = F_f(x) + 1 \quad \text{for all } x \in \mathbb{R}.$$

This periodic extension allows the study of the classical rotation number of the associated monotonic map.

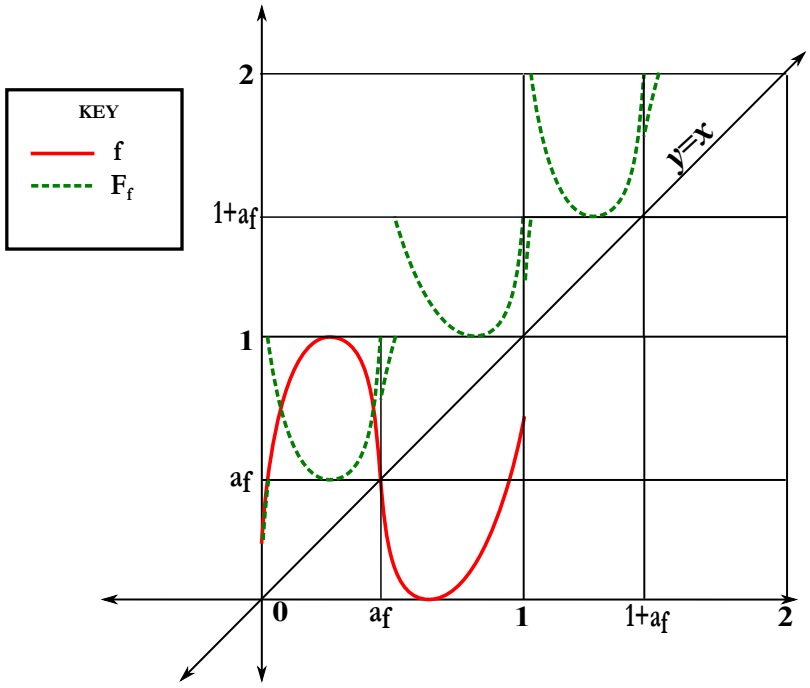

Figure 5.3: The extension of F_f to $\mathbb{R}$ as a degree-one lift.

5.10 Construction of the Monotonic Map G_f

The function G_f is defined as the lower envelope of F_f:

$$G_f(x) = \inf\{F_f(y) \mid y \geq x\}.$$

It is continuous and non-decreasing, and its rotation number reflects the minimal long-term displacement under iteration.

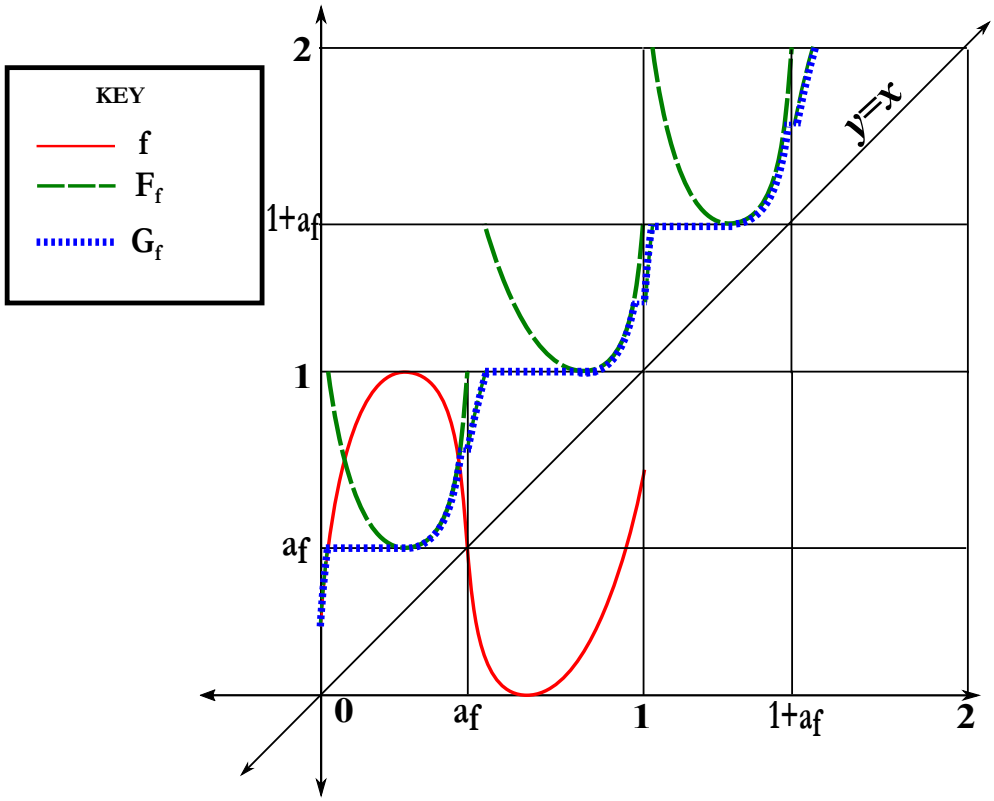

Figure 5.4: The function G_f, the lower bound of F_f, used to compute the rotation number r_f.

5.11 Interpretation of the Rotation Number

The rotation number of the function G_f, given by:

$$\rho(G_f) = \lim_{n\to\infty} \frac{G_f^n(x) - x}{n},$$

exists and is independent of $x \in \mathbb{R}$. This number equals the left endpoint r_f of the over-rotation interval of the original map f. Therefore, the computation of the classical rotation number of G_f gives an effective algorithm to determine r_f.

$$r_f = \rho(G_f).$$

This link between symbolic over-rotation behavior and classical rotation theory provides deep insight into the global dynamics of interval maps.

5.12 Applications

The theory of over-rotation numbers and intervals for continuous interval maps, as developed in this chapter, has significant implications both within dynamical systems theory and in broader scientific contexts. Below, we highlight several key areas where the ideas presented here play a fundamental role.

- **Classification of Periodic Orbits:** The over-rotation number provides a refined tool for distinguishing between different periodic orbits of a map. In particular, the over-rotation interval $I_f = [r_f, \frac{1}{2}]$ encapsulates global information about the distribution and density of periodic points, enabling classification of the system's long-term behaviors.

- **Detection of Chaos:** A wide over-rotation interval (i.e., with r_f close to 0) often indicates high dynamical complexity and chaotic behavior. This makes the computation of r_f a diagnostic tool for identifying systems with sensitive dependence on initial conditions and a rich set of periodic dynamics.

- **Numerical Approximation of Dynamical Invariants:** The algorithm described for computing r_f using the lift G_f gives a concrete method to estimate rotation-theoretic invariants in practical settings. This is particularly useful for analyzing data from simulations of interval maps in applied problems, including population dynamics and control theory.

- **Symbolic Dynamics and Kneading Theory:** Over-rotation numbers can be linked with symbolic sequences (itineraries), allowing them to serve as a bridge between symbolic dynamics and geometric behavior. This connection plays an important role in kneading theory and helps to construct kneading invariants for unimodal and multimodal maps.

- **Applications in One-Dimensional Real Models:** Many natural and engineered systems are modeled by one-dimensional continuous maps. Examples include logistic-type population models, heartbeat models in physiology, and electronic circuits. The tools developed in this chapter can be directly applied to analyze and classify the behaviors of such systems.

- **Foundational Step for Discontinuous Maps:** The insights gained from the over-rotation theory for continuous maps serve as a theoretical foundation for the extension to discontinuous maps, as discussed in later chapters. The lifting method and rotation number framework introduced here are adapted to more complex settings involving piecewise-defined and non-continuous maps.

5.13 Conclusion

In this chapter, we explored the rotation theory for continuous interval maps, with a particular emphasis on the concept of *over-rotation numbers*. Beginning with a general definition inspired by symbolic dynamics, we extended the classical notion of rotation number to interval maps through a carefully constructed function χ, capturing the turning behavior of orbits relative to fixed points.

We discussed how the over-rotation number of a periodic orbit is derived and introduced the over-rotation interval $I_f = [r_f, \frac{1}{2}]$ as a compact descriptor of the dynamical richness of the system. A key result highlighted that the presence of a periodic orbit with a given over-rotation number implies the existence of orbits with all lesser rational over-rotation numbers, establishing a dense and ordered structure within the interval.

A significant portion of the chapter was devoted to describing an algorithm for computing the left endpoint r_f of the over-rotation interval. This method leverages a conjugacy of the original map to construct a monotonic degree-one lift G_f, whose classical rotation number equals r_f. Through this transformation pipeline—from f to g_f, to F_f, and finally to G_f—we bridge the symbolic over-rotation framework with the classical rotation theory on the circle.

The algorithm provides not only a computational tool but also a conceptual unification of symbolic and geometric aspects of dynamical systems. The inserted graphs throughout the section visualize each stage of this reduction and clarify the structure of the associated maps.

Overall, this approach enhances our understanding of interval dynamics by revealing how local turning behaviors accumulate into global rotational patterns, ultimately offering a powerful lens through which to study the complexity of continuous interval maps.

Chapter 6

Rotation Theory for Some Class of Discontinuous Maps of Any Modality

In this chapter, we explore the rotation theory associated with discontinuous maps, particularly those satisfying a specific structural condition referred to as the **N-type condition**. We aim to construct and analyze certain auxiliary maps, ultimately leading to the definition of an over-rotation interval for such maps.

6.1 Preliminaries and Definitions

Let $f : [0, 1] \to [0, 1]$ be a discontinuous map satisfying the **N-type condition**, defined as follows:

Let M_f denote the point of maximum of f, m_f the point of minimum, and a_f a fixed point of f. The function f satisfies the *N-type condition* if:

1. $M_f \leq a_f \leq m_f$,
2. $f(x) \leq a_f$ for all $x \in [a_f, m_f]$ and $f(x) \geq a_f$ for all $x \in [M_f, a_f]$.

We consider such a function $f : [0, 1] \to [0, 1]$ that is discontinuous and satisfies this condition.

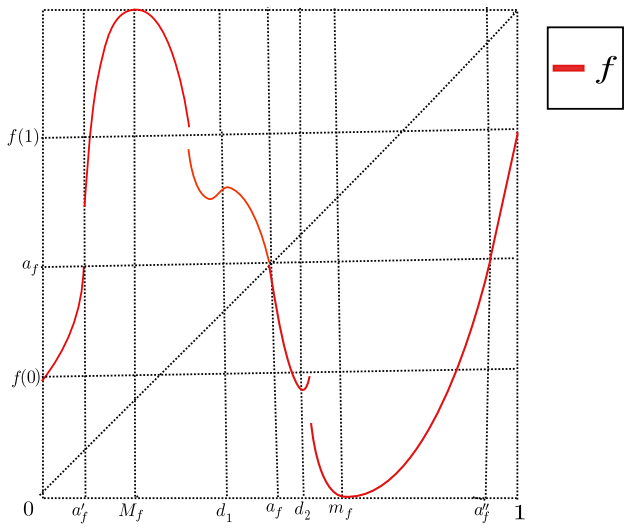

Figure 6.1: **graph of** f .

6.2 Construction of the Involution Map σ_f

We define a map $\sigma_f : [0,1] \to [0,1]$, which is bijective and self-inverse:

$$\sigma_f(x) = \begin{cases} x & \text{if } x \in [0, a_f], \\ 1 + a_f - x & \text{if } x \in [a_f, 1]. \end{cases}$$

6.3 Definition of the Modified Map g_f

We now define the map $g_f : [0,1] \to [0,1]$ by conjugation:

$$g_f = \sigma_f \circ f \circ \sigma_f.$$

That is, $g_f(x)$ is defined piecewise as follows (where $a'_f, a''_f \in [0,1]$):

$$g_f(x) = \begin{cases} f(x) & \text{if } x \in [0, a'_f], \\ 1 + a_f - f(x) & \text{if } x \in [a'_f, a_f], \\ f(1 + a_f - x) & \text{if } x \in [a_f, a''_f], \\ 1 + a_f - f(1 + a_f - x) & \text{if } x \in [a''_f, 1]. \end{cases}$$

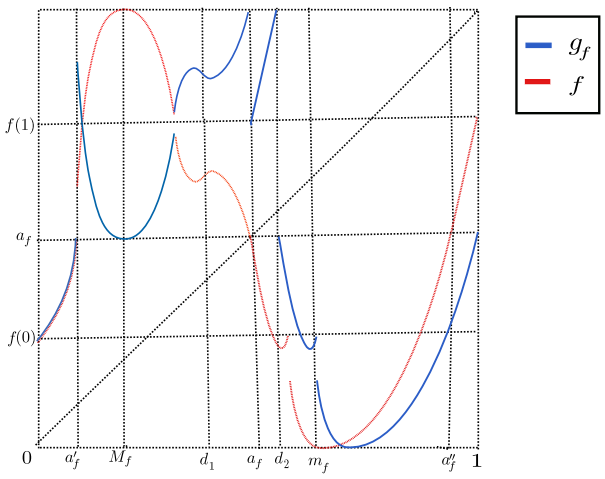

Figure 6.2: graph of g_f

6.4 Definition of the Liftin Map F_f

We now define a lifted function $F_f : [0,1] \to [0,2]$ as follows:

$$F_f(x) = \begin{cases} g_f(x) + 1 & \text{if } x > a_f \text{ and } f(x) < a_f, \\ g_f(x) & \text{otherwise.} \end{cases}$$

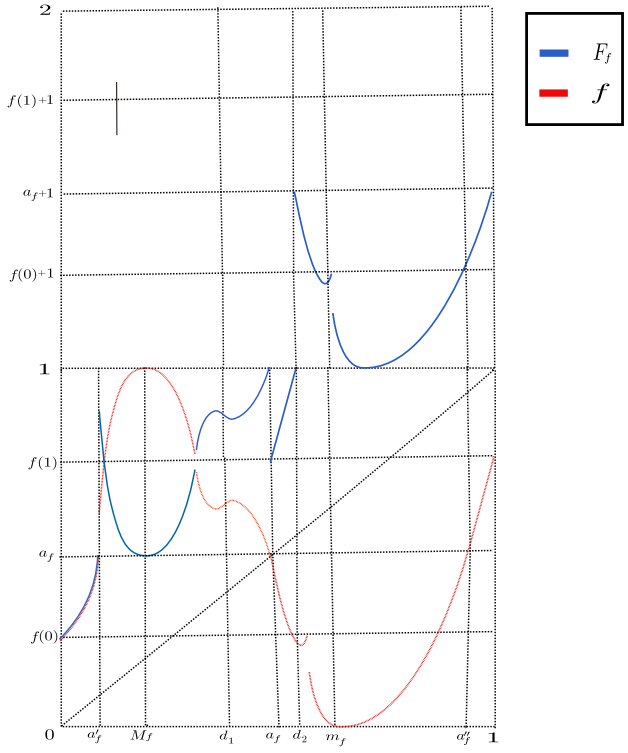

Figure 6.3: graph of F_f

We extend F_f to the real line by periodicity:

$$F_f(x+1) = F_f(x) + 1, \quad \text{for all } x \in \mathbb{R}.$$

6.5 Lower Bound Map F_l

Define the function $F_l : \mathbb{R} \to \mathbb{R}$ as:

$$F_l(x) = \inf\{F_f(y) \mid y \geq x\}.$$

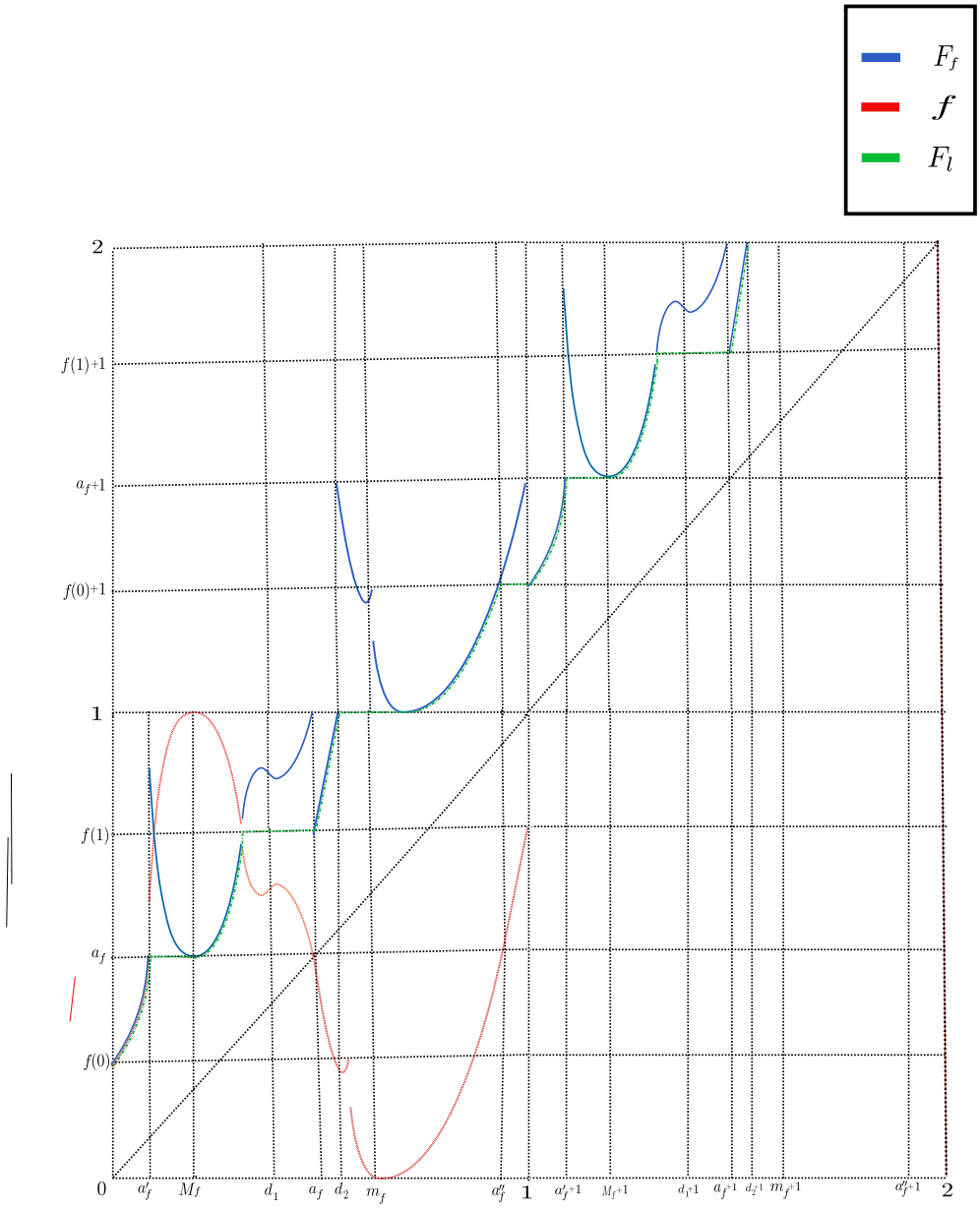

Figure 6.4: graph of F_f and F_l

6.6 Rotation Number and Over-Rotation Interval

Although the map F_f is discontinuous and classical tools for degree-one circle maps may not apply, the function F_l is continuous and non-decreasing. Therefore, it admits a well-defined rotation number, denoted by ρ, which is constant across all points.

We aim to show that the over-rotation interval of the map f is given by $[\rho, \frac{1}{2}]$.

6.7 Monotonicity Argument

By construction, we have:

$$F_l(x) \leq F_f(x) \quad \text{for all } x \in \mathbb{R}.$$

Since F_l is monotonic:

$$F_l(F_l(x)) \leq F_l(F_f(x)) \leq F_f(F_f(x)).$$

So,

$$F_l^2(x) \leq F_f^2(x).$$

By induction, for all $n \in \mathbb{N}$:

$$F_l^n(x) \leq F_f^n(x).$$

Taking limits:

$$\lim_{n\to\infty} \frac{F_l^n(x)}{n} \leq \lim_{n\to\infty} \frac{F_f^n(x)}{n}.$$

Thus, the rotation number of each point under F_l is less than or equal to the rotation number of the corresponding point under F_f.

Since the rotation number under F_f, when it exists, corresponds to the over-rotation number of f, we conclude:

$$\rho \leq \text{over-rotation number of } f.$$

Hence, the over-rotation interval of f is:

$$[\rho, \tfrac{1}{2}].$$

□

6.8 Applications of Rotation Theory for Discontinuous Maps

Beyond the mathematical context of interval exchange transformations, rotation theory for discontinuous maps finds relevance in a variety of applied disciplines. Some notable examples include:

- **Piecewise-defined economic or biological models**: In systems where evolution is governed by rules that change abruptly based on thresholds or states (e.g., predator-prey models with harvesting, or taxation schemes in economics), discontinuities naturally occur. Rotation theory helps in understanding the long-term average behavior of such systems.
- **Hybrid systems and control theory**: These systems combine discrete and continuous components, often switching dynamics based on logical conditions. Rotation-like quantities can describe periodic switching behavior and help in designing stable control strategies.
- **Digital signal processing**: In areas involving quantization, sampling, or switching circuits, underlying maps may exhibit discontinuities. The analysis of their rotation behavior aids in predicting frequency content and aliasing effects.
- **Computer graphics and visualization**: Iterated function systems and texture mapping often rely on transformations that are not continuous everywhere. Understanding their rotation structure can inform stability and aesthetic quality of generated patterns.

These applications emphasize the broader importance of extending classical rotation theory into discontinuous regimes, offering powerful tools for analyzing both natural and engineered systems.

6.9 Conclusion

In this chapter, we developed a framework for analyzing the rotational dynamics of a class of discontinuous maps on the unit interval that satisfy the *N-type condition*. By constructing the auxiliary involution map σ_f and the conjugated map g_f, we laid the groundwork for defining a lifted map F_f that encodes the over-rotation dynamics of the original map f.

To address the discontinuity in F_f, we introduced the lower bound map F_l, which is both continuous and monotone. This allowed us to rigorously define a rotation number ρ, and to establish a containment relationship between the iterates of F_l and F_f. Through a careful monotonicity argument, we concluded that the over-rotation interval for f is bounded below by ρ and above by $\frac{1}{2}$.

This analysis not only extends classical rotation theory to a broader class of maps but also illustrates how structural symmetries and lifting techniques can be leveraged to recover rotational information in settings where continuity is absent. The over-rotation interval $[\rho, \frac{1}{2}]$ encapsulates the essence of rotational behavior in this discontinuous context, and may serve as a basis for further investigations into periodicity, entropy, and bifurcation structures in discontinuous dynamical systems.

Chapter 7

Applications

7.1 Introduction

Rotation theory has proven to be a powerful framework for understanding the long-term behavior of points under iteration in dynamical systems. Traditionally applied to continuous maps of the circle or the interval, it can also be adapted to certain classes of discontinuous maps. In this final chapter, we explore the application of rotation theory to *interval exchange transformations* (IETs)—a class of piecewise isometries defined on intervals that are naturally discontinuous at a finite number of points.

Our goal is to illustrate how rotation-theoretic tools, such as rotation numbers and intervals, extend meaningfully to IETs. This enables a refined understanding of their dynamics, such as minimality, periodicity, and ergodic behavior.

An **interval exchange transformation** (IET) is a bijective map defined on an interval, constructed by partitioning the interval into a finite number of subintervals, and permuting these subintervals via translations.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a vector of positive real numbers such that $\sum_{i=1}^{n} \lambda_i = 1$, and let π be a permutation of $\{1, 2, \ldots, n\}$. Define the subintervals:

$$I_i = \left[\sum_{j=1}^{i-1} \lambda_j, \sum_{j=1}^{i} \lambda_j \right), \quad \text{for } i = 1, \ldots, n.$$

An *IET* $T : [0, 1) \to [0, 1)$ is defined by translating each I_i to $J_{\pi(i)}$ via:

$$T(x) = x - \sum_{j=1}^{i-1} \lambda_j + \sum_{j=1}^{\pi(i)-1} \lambda_j, \quad \text{for } x \in I_i.$$

Each T is piecewise isometric (preserving distances within each subinterval), discontinuous at a finite set of endpoints, and bijective on its domain.

Despite being globally discontinuous, IETs preserve many regular dynamical properties, such as measure preservation and zero topological entropy. The number of subintervals n serves as an analog of modality for these maps.

7.2 Rotation Theory Perspective on IETs

While classical rotation theory focuses on circle maps of degree one, we can draw analogies between IETs and circle rotations, especially when $n = 2$. In fact, every irrational rotation on the circle can be modeled by a 2-interval exchange transformation.

7.2.1 Case $n = 2$: Circle Rotations as IETs

Let $\alpha \in (0, 1)$ be irrational. Define the IET on two intervals:

$$I_1 = [0, 1 - \alpha), \quad I_2 = [1 - \alpha, 1),$$

$$T(x) = \begin{cases} x + \alpha & \text{if } x \in I_1, \\ x + \alpha - 1 & \text{if } x \in I_2. \end{cases}$$

This map is equivalent to a rotation $R_\alpha : x \mapsto x + \alpha \mod 1$. The rotation number of T is α, and since α is irrational, the orbit of every point is dense in $[0, 1)$.

7.2.2 General n: Interval Exchanges and Rotation Vectors

For $n \geq 3$, IETs exhibit more complex behavior. Rotation numbers generalize to **rotation vectors**, reflecting the long-term frequencies with which orbits visit the various subintervals. These are typically defined via Birkhoff averages:

$$\rho_i(x) = \lim_{n \to \infty} \frac{1}{n} \#\{0 \leq k < n : T^k(x) \in I_i\},$$

the symbol $\#$ denotes cardinality—that is, the number of elements in a set. assuming the limit exists.

The vector $\rho(x) = (\rho_1(x), \ldots, \rho_n(x)) \in \mathbb{R}^n$ represents the proportion of time that the orbit of x spends in each subinterval.

Under suitable assumptions (e.g., minimality, ergodicity), this limit exists and is constant for almost every point x. In this sense, the rotation vector encodes a global statistical property of the system.

7.3 Minimality and Over-Rotation Phenomena

An IET is said to be *minimal* if every orbit is dense in $[0, 1)$. This depends on both the length vector λ and the permutation π. For irreducible permutations and irrational parameters, the system is typically minimal and uniquely ergodic, and the rotation vector describes the unique invariant measure.

While classical over-rotation intervals are not defined in IETs in the same way as for continuous interval maps, one can still define analogues via comparing orbit complexity or defining "skew" maps that mix translations and reflections.

Some IETs can be "suspended" to flows on surfaces, leading to geometric interpretations of rotation numbers and intervals as periods of orbits on translation surfaces.

7.4 Applications and Connections

IETs appear naturally in several areas of mathematics:

- **Billiards in Rational Polygons:** The motion of a billiard ball inside a rational polygon unfolds into an IET.
- **Symbolic Dynamics:** IETs serve as models for minimal subshifts with low complexity.
- **Teichmüller Dynamics:** The study of translation surfaces often involves flows whose Poincaré return maps are IETs.
- **Ergodic Theory:** IETs provide examples of systems that are measure-preserving but not mixing, and that exhibit unique ergodicity.

Rotation theory helps in understanding long-term statistical behavior, invariant measures, and recurrence properties of these systems.

7.5 Conclusion

This chapter has outlined the connections between rotation theory and interval exchange transformations, a rich class of discontinuous maps with far-reaching applications. While rotation numbers in their classical form may not directly apply, generalizations such as rotation vectors and symbolic codings enable a fruitful adaptation of rotation-theoretic ideas.

In the case of 2-interval IETs, we recover classical irrational rotations. For higher n, IETs model more intricate behaviors, yet still permit a statistical analysis via rotation vectors. This illustrates the robustness of rotational ideas beyond the realm of continuous dynamics, extending into the study of discontinuous yet highly structured transformations.

These insights not only demonstrate the flexibility of rotation theory but also suggest new avenues for its application in symbolic, geometric, and ergodic contexts involving discontinuous maps.

Chapter 8

Future Work

8.1 Rotation Theory for Billiard Systems

In this thesis, we developed a rotation theory for a class of discontinuous maps satisfying an N-type condition, and demonstrated how this framework extends naturally to certain classes of interval exchange transformations (IETs). A promising direction for future research is to further explore the connection between rotation theory and IETs, particularly in the context of billiard systems. A *billiard system* consists of a point mass moving freely within a bounded polygonal domain, undergoing elastic reflections upon contact with the boundary. Since the dynamics of billiards in rational polygons can often be modeled using IETs, developing a rotation theory tailored to such systems could provide new insights into the structure of their reflective trajectories. This approach may offer a novel way to characterize orbit types, recurrence properties, and directional behaviors, thereby deepening the connection between discrete dynamical systems and geometric applications.

8.2 Computational Tools for Over-Rotation Theory

Another natural extension of this work is the implementation of computational tools to complement the theoretical framework. In particular, designing and developing efficient Python algorithms to compute the over-rotation interval for discontinuous maps satisfying the N-type condition would significantly enhance the accessibility and applicability of the theory. Such algorithms could facilitate experimentation, visualization, and validation of theoretical results, as well as sup-

port investigations in applied contexts, including engineering systems and mathematical models involving discontinuous dynamics. This computational perspective would not only strengthen the practical impact of the theory but also potentially reveal new patterns and behaviors that are not easily uncovered through analytical techniques alone.

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